

Classical multi-party signaling can maximally violate predefined causal order

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A framework for macroscopic correlations where the notion of temporal ordering is not considered as fundamental is constructed by taking the macroscopic (classical) limit of a recent framework for quantum correlations by Oreshkov, Costa, and Brukner [Nat. Commun. **3**, 1092 (2012)]. Because the same holds for *two parties*, it was conjectured that also in the *multi-party* setting, a predefined causal order emerges in that limit. We show that counter to this belief, the framework for macroscopic correlations *does* allow for deterministic signaling between three or more parties incompatible with any predefined causal order.

INTRODUCTION

Non-local correlations, as predicted by quantum theory and observed in nature [1], refute local realism [2] if we assume the common world view that spatial and temporal separation are fundamental, and that spatially separated settings can be chosen independently. With this refutation of local realism we lose some explanatory power of quantum theory beyond its mathematical formalism. Already Einstein, Podolsky and Rosen asked that a theory should give a local description of physical reality and concluded that quantum theory, under this condition, is incomplete [3]. A promising idea in finding an explanatory description of the predicted correlations is to consider time and space not as fundamental, but rather as emerging from fundamental principles. A step into this direction was initiated by Hardy in his program of merging the theory of general relativity with quantum theory [4], by proposing to extend quantum theory to dynamical causal structures. Oreshkov, Costa, and Brukner developed a framework for quantum correlations [5] based on this idea, where they drop the assumption of a global background time, while keeping the assumptions that locally, nature is described by quantum theory, and no logical paradoxes arise. Interestingly, in a scenario with two parties, if we consider the macroscopic limit of the quantum systems, i.e., enforce both parties' physics to be described by classical probability theory instead of quantum theory, a predefined causal order emerges, suggesting that also for any number of parties a fixed temporal ordering might result in the macroscopic limit.

This work studies the framework for correlations with no causal order in the macroscopic limit where locally, nature is described by classical probability theory. We show that for three parties or more, deterministic classical correlations incompatible with any predefined causal order arise, and thus, a predefined causal order does not emerge in that limit.

For that purpose we continue by defining causality, fol-

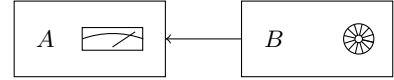


Figure 1. If party B can freely choose a variable (here, visualized by a knob), and party A can read off a quantity that is correlated to B 's variable, then B can signal to A , implying that A is in the causal past of B ($A \succeq B$).

lowed by a description of a causal game, with an upper bound on the game winning probability in a world with predefined causal order. Thereafter we develop the macroscopic framework for correlations with no causal order, and show how it can be used to perfectly win the causal game for any number n of parties, where n is larger than two.

CAUSAL ORDER

When observing a physical process we can ultimately speak only of the measurements we chose to make, and of their outcomes. Therefore, physical systems can be thought of as black boxes that we probe. Taking this perspective, we describe all *physical quantities*, in the following called *quantities*, as functions of *variables*. Each party A is described by a set of *variables* $A = \{a_i\}_I$ that are chosen *freely* by the party (we will not give a definition of *free choice*, rather we define each *variable* as being *intrinsically free*) and by a set of *quantities* $X = \{x_j\}_J$ the party can access. Quantities are therein deterministic functions of variables. Probabilistic quantities are derandomized deterministic functions, where the random essence is simply another variable. We say a quantity x_j as a function of a variable a_k , *causally depends* on a_k , and is in the *causal future* of a_k , denoted by $a_k \preceq x_j$ or $x_j \succeq a_k$. Conversely, the variable a_k is in the *causal past* of x_j . The negation of these relations are denoted by $\not\succeq$ and $\not\preceq$. This definition does not induce a causal order between quantities, between variables, nor between any quantity and the variables it does not depend on.

Let us introduce a second party B described by the set of variables $B = \{b_k\}_K$, with access to the quantities $Y = \{y_\ell\}_L$. Now, quantities could depend on variables from both parties. If, for instance, party A has a quantity that depends on a variable of B , then we say B can *signal* to A (see Figure 1). In the remaining part of this work we will assume unidirectional signaling, e.g., if A can signal to B , then B cannot signal to A . This allows us to causally order parties. If at least one quantity of A depends on a variable of B , but no quantity of B depends on any variable of A (which is the condition for unidirectional signaling), then A is in the causal past of B . Formally, if $\exists x \in X, b \in B$, such that $b \preceq x$, and $\forall y \in Y, a \in A, a \not\preceq y$, then $B \preceq A$.

Consider a two-party scenario with parties A, B , each having a respective single variable a, b and a respective single quantity x, y . We call a theory *causal realistic* if and only if all achievable probability distributions $\Pr(x, y|a, b)$, which we call *processes*, result from a convex mixture of possible causal orders, i.e.,

$$\begin{aligned} \Pr(x, y|a, b) = & \Pr(\alpha) \sum_{\lambda} \Pr(\lambda|\alpha) \Pr(x|a, \lambda, \alpha) \Pr(y|a, b, \lambda, \alpha) \\ & + (1 - \Pr(\alpha)) \sum_{\lambda} \Pr(\lambda|\alpha) \Pr(x|a, b, \lambda, \alpha) \Pr(y|b, \lambda, \alpha), \end{aligned}$$

where α is the event $A \preceq B$ and λ is a shared quantity depending on a random variable neither chosen by A nor by B . The *realism* attribute of a causal realistic theory is that the causal order is *predefined*. For more than two parties, the definition of causal realism becomes more subtle: Suppose we have three parties A, B , and C , and A is in the causal past of B and C . Then A can specify the causal order between B and C , while the causal order remains predefined [6]. Hence, the definition of an n -party causal realistic theory with parties A_1, \dots, A_n , respective variables a_1, \dots, a_n , and respective quantities x_1, \dots, x_n , reads

$$\Pr(x_1, \dots, x_n|a_1, \dots, a_n) = \sum_i \Pr(\alpha_i) \Pr_i(x_1, \dots, x_n|a_1, \dots, a_n, \alpha_i),$$

where α_i is the event that party A_i precedes the others, i.e., $\bigwedge_{j \neq i} A_i \preceq A_j$, and $\Pr_i(x_1, \dots, x_n|a_1, \dots, a_n, \alpha_i)$ is the process compatible with the causal structure where A_i is first. In order for $\sum_i \Pr(\alpha_i)$ to sum up to unity, we assign all causal structures where multiple parties A_{i_1}, \dots, A_{i_m} are first to the event $\alpha_{\min(i_1, \dots, i_m)}$.

CAUSAL GAME

The following multi-party game cannot be won in a causal realistic scenario. Denote by A_1, \dots, A_n the n par-

ties participating in the game. Each party A_i has a uniformly distributed binary variable a_i and access to the binary quantity x_i , and to the shared quantity $m = x_0$, which is a function of a variable x_0 uniformly distributed on the range $1, \dots, n$ that belongs to a dummy party (we just need it as a resource of shared randomness). One can think of the party's variable a_i as its input, and the party's quantity x_i as its output. For a given m , the game is won whenever party A_m outputs x_m as the parity of the inputs to all other parties, i.e., $x_m = \bigoplus_{i \neq m} a_i$. Therefore, the success probability of winning the game is

$$p_{\text{succ}} := \frac{1}{n} \sum_{m=1}^n \Pr \left(x_m = \bigoplus_{i \neq m} a_i \middle| m \right).$$

In a causal realistic setup, this success probability is upper bounded by $(2n - 1)/2n$. To see this, note that if A_i is first, then it will remain first. However, for $n \geq 3$, the last party can be specified by A_i . Thus, all but one summands (where the first party itself has to guess the parity of the others' inputs) of the success probability expression are 1, the remaining summand is $1/2$, as the output x_i of A_i will match the desired bit with a chance of $1/2$. By repeating the experiment m times, where m grows asymptotically faster than n , e.g., $m = n^2$, one can bring the winning probability arbitrarily close to zero.

FRAMEWORK FOR MACROSCOPIC CORRELATIONS WITH NO CAUSAL ORDER

Instead of assuming that locally, nature is described by quantum theory [5], we take the macroscopic limit of the systems, and thus assume that locally, nature is described by *classical probability theory*. Next to this assumption, we ask the probabilities of the outcomes to be non-negative and to sum up to unity, which forbids causal loops [5]. Suppose each party has a closed laboratory, meaning they do not interact, and each laboratory is opened once, during which the only interaction happens. When a laboratory is opened, it receives a state, manipulates it, and outputs a state. Thus, in the setting with local classical probability theory, such a laboratory is described by a conditional probability distribution $P_{O|I}$, where I is the input space to the laboratory, and O is the output space of the laboratory.

Let us consider the parties as described in the causal game. We denote the input space of party A_i with I_i , and its output space with O_i . Therefore, the i th local laboratory is described by the distribution $P_{X_i, O_i|A_i, I_i}$, where X_i is the space of its quantity x_i , and A_i is the space of its variable a_i . As we do not have other global assumptions than that the overall picture should describe a probability distribution, we can describe everything outside the laboratories by the distribution $P_{I_1, \dots, I_n|O_1, \dots, O_n}$, which we call W (see Figure 2),

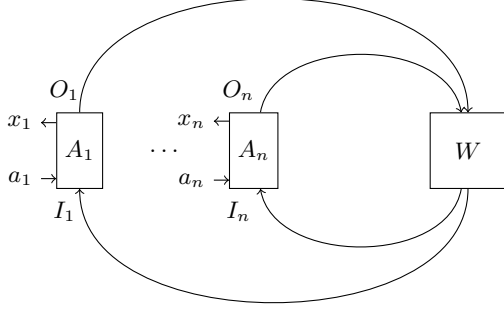


Figure 2. Party A_i is described by $P_{X_i, O_i | A_i, I_i}$, and its output on space O_i is fed to W , describing everything outside the laboratories, which therefore also provides the input to space I_i . The box W is described by $P_{I_1, \dots, I_n | O_1, \dots, O_n}$.

with the restriction that for any choice of a_1, \dots, a_n , i.e., for any $P_{X_1, O_1 | I_1}, \dots, P_{X_n, O_n | I_n}$, the global probability distribution $P_{X_1, O_1 | I_1} \cdots P_{X_n, O_n | I_n} \cdot P_{I_1, \dots, I_n | O_1, \dots, O_n}$ sums up to unity, and in particular, the distribution $P_{O_1 | I_1} \cdots P_{O_n | I_n} \cdot P_{I_1, \dots, I_n | O_1, \dots, O_n} = P_{O_1, I_1, \dots, O_n, I_n}$ sums up to unity.

To tackle this restriction formally, we represent a probability distribution P_X as a real, positive, diagonal matrix, having unit trace, with the diagonal entries $P_X(x)$. We use the symbol \hat{P}_X to denote such a matrix. A conditional probability distribution $P_{X|Y}$ is a collection of non-conditional probability distributions for each value of $y \in Y$. Thus, we represent $P_{X|Y}$ equivalently but with trace $|Y|$, and use the symbol $\hat{P}_{X|Y}$. The condition that for a fixed $y \in Y$ the probability $P_{X|Y}(x, y)$ sums up to unity is reflected in the condition that, if we trace out X from the matrix $\hat{P}_{X|Y}$, we are left with the identity matrix. The product of two distributions P_X and P_Y is defined by all combinations of $x \in X$ and $y \in Y$. Thus, the product in the matrix representation corresponds to the tensor product denoted by \otimes . To obtain the marginal distribution from a joint distribution we use the partial trace. From this follows that the output state of a laboratory $P_{O_i | I_i}$ given the input state P_{I_i} is $\text{Tr}_{I_i}(\hat{P}_{O_i | I_i} \cdot (\hat{P}_{I_i} \otimes \mathbb{1}_{O_i}))$, where $\mathbb{1}_{O_i}$ is the identity matrix. This ultimately allows us to use the framework of Oreshkov, Costa, and Brukner [5], restricting ourselves to diagonal matrices, i.e., all objects (W and local operations) are simultaneously diagonalizable in the computational basis, and thus can be expressed using the identity matrix $\mathbb{1}$ and the Pauli z matrix σ_z . In order to avoid causal loops, we know from their framework [5] that, if we express $P_{I_1, \dots, I_n | O_1, \dots, O_n}$ as a matrix $W = c \sum_i o_i$, where c is a constant depending on the dimension of W and $o_i = R_i^{I_1} \otimes S_i^{O_1} \otimes \cdots \otimes R_i^{I_n} \otimes S_i^{O_n}$, where the superscripts denote the respective spaces, then o_i must either be the identity matrix, or have at least one party

with identity on the output space and a traceless part on the input space, i.e., $\forall i$, either $o_i = \mathbb{1}$ or, $\exists j$, such that $S_i^{O_j} = \mathbb{1}$ and $\text{Tr}(R_i^{I_j}) = 0$.

WINNING THE CAUSAL GAME PERFECTLY

In order to win the causal game using this framework, we need to provide the distribution $P_{I_1, \dots, I_n | O_1, \dots, O_n}$ and all local distributions describing the laboratories. For that purpose we use that if a set $\{o_i\}_I$ of matrices with ± 1 eigenvalues together with the matrix multiplication form an Abelian group, then $\sum_i o_i$ is a positive semi-definite matrix. An Abelian group is closed, i.e., $\forall i, j \in I, \exists k \in I$, such that $o_i \cdot o_j = o_k$, is associative, has an identity element o_0 , i.e., the identity matrix, each element has an inverse, and all elements mutually commute. To prove positivity, take the eigenvector $|v\rangle$ that has the smallest eigenvalue λ_{\min} , i.e., $\sum_i o_i |v\rangle = \sum_i \lambda_i^v |v\rangle = \lambda_{\min} |v\rangle$, where λ_i^v is the eigenvalue of o_i with respect to the eigenvector $|v\rangle$. Because all elements mutually commute, they share a common set of eigenvalues, and thus, the eigenvalues of the sum of the elements are the sum of the eigenvalues of the single elements. Let o_i be an element contributing negatively to λ_{\min} , i.e., $o_i |v\rangle = -|v\rangle$. As the set forms a closed group, for every $j \neq 0$, there exists a $k \neq j$, such that $o_i \cdot o_j = o_k$. From this follows $-\lambda_j^v = \lambda_k^v$, implying $\sum_i \lambda_i^v = 0$.

We construct the distribution $P_{I_1, \dots, I_n | O_1, \dots, O_n}$ for n odd and larger than two. Let $\{o_i\}_I$ be the set of all matrices that can be written as $o_i = M_{i,1} \otimes M_{i,2} \otimes \cdots \otimes M_{i,n}$, where $M_{i,j} \in \{\mathbb{1}, \sigma_z\}$, with an even number of σ_z 's for each $i \in I$. From the equality $\sigma_z \cdot \sigma_z = \mathbb{1}$ follows that for any $i, j \in I$, the product $o_i \cdot o_j$ again has an even number of σ_z 's, and thus is an element in the set, forming a closed group. Furthermore, all elements mutually commute, have ± 1 eigenvalues, and each element is its own inverse. Thus, their sum is a positive semi-definite matrix. The distribution $P_{I_1, \dots, I_n | O_1, \dots, O_n}$ as a matrix W_n is built by taking the sum over all group elements, where the matrix $M_{i,j}$ of the group element o_i is associated to the input space of the j th party, i.e., I_j , and to the output space of the $((j-1) \bmod n)$ th party, i.e., $O_{(j-1) \bmod n}$,

$$W_n = \frac{1}{2^n} \sum_i o_i^{I_1, I_2, \dots, I_n} \otimes o_i^{O_2, O_3, \dots, O_n, O_1}.$$

By construction, W_n is positive semi-definite, reflecting that all probabilities are positive. Each group element o_i for $i > 0$ contains at least one $\mathbb{1}$ matrix preceded by a σ_z matrix. Let such a σ_z matrix of o_i be at position j , i.e., $M_{i,j} \otimes M_{i,(j+1) \bmod n} = \sigma_z \otimes \mathbb{1}$, then the j th party has identity on the output and a traceless part on the input space, which is a requirement for the summand to be valid. This furthermore implies that tracing out the spaces I_1, \dots, I_n from W_n yields the identity matrix,

reflecting that for each condition (input to W_n) the probabilities sum up to unity.

This construction, however, works only if n is odd and larger than two, because, for n even, the group would contain the element $\sigma_z^{\otimes n}$, leading to a causal loop. For this last case, we enlarge one input and one output space, and construct the distribution as follows. Let $\{o_i\}_I$ be the group used to construct W_n for $n - 1$ parties. The set for n parties is $\{o'_i \otimes o_i\}_I \cup \{\bar{o}'_i \otimes \bar{o}_i\}_I$, where o'_i are the first two 2×2 matrices of o_i , i.e., $o'_i = M_{i,1} \otimes M_{i,2}$, and \bar{o}_i are the o_i elements with $\mathbb{1}$ and σ_z exchanged, i.e., $\bar{o}_i = o_i \cdot \sigma_z^{\otimes n-1}$. This set forms an Abelian group, as all required properties follow immediately from $\{o_i\}_I$.

The distribution $P_{I_1, \dots, I_n | O_1, \dots, O_n}$ as a matrix W_n is then constructed as before, with the exception that o'_i is considered as single submatrix, i.e., o'_i is associated to the input space of the first party, and to the output space of the n th party,

$$W_n = \frac{1}{2^{n+1}} \sum_i \left(o'^{I_1}_i \otimes o^{I_2, \dots, I_n}_i \otimes o^{O_1, \dots, O_{n-1}}_i \otimes o'^{O_n}_i \right. \\ \left. + o'^{I_1}_i \otimes \bar{o}^{I_2, \dots, I_n}_i \otimes \bar{o}^{O_1, \dots, O_{n-1}}_i \otimes o'^{O_n}_i \right).$$

The first party has a two-bit input and the n th party has a two-bit output. Again, by construction, W_n is positive semi-definite. Each summand different from identity contains at least one identity submatrix preceded by a traceless submatrix, fulfilling the requirements that only valid summands appear, and that under each condition, the probabilities sum up to unity.

In the n odd scenario, the local distribution for the i th party, given m , to win the game is

$$\left(\frac{\mathbb{1} + (-1)^{x_i} \sigma_z}{2} \right)^{I_i} \otimes \left(\frac{\mathbb{1} + (-1)^{a'_i} \sigma_z}{2} \right)^{O_i},$$

where $a'_i = a_i$ for $i = (m+1) \bmod n$, and $a'_i = a_i + x_i$ otherwise. The strategies for n even are equivalent, except that A_1, A_n , has a two-bit input space, respectively output, space, and depending on m , uses the first, second, or both (with the parity as the relevant bit) bit-spaces to receive, send the desired bit. All local operations are classical since they are diagonal, i.e., consist only of measuring and preparing states in the σ_z basis.

Let P_i be the operation of the i th party. The distribution $\Pr(x_m | a_1, \dots, a_n, m)$ is

$$\Pr(x_m | a_1, \dots, a_n, m) = \sum_{x_{i \neq m}} \Pr(x_1, \dots, x_n | a_1, \dots, a_n, m) \\ = \sum_{x_{i \neq m}} \text{Tr}((P_1 \otimes \dots \otimes P_n) \cdot W_n) \\ = \frac{1}{2} \left(1 + (-1)^{x_m + \sum_{i \neq m} a_i} \right).$$

This is because, when summing over all elements of W_n , each term except 1 and $(-1)^{x_m + \sum_{i \neq m} a_i}$ is either zero

or depends on a variable $x_{i \neq m}$, which, in the process of marginalization over $x_{i \neq m}$, cancels. For each m , the winning probability is $\Pr(x_m = \bigoplus_{i \neq m} a_i | m) = 1$. Therefore, the game is won with certainty.

CONCLUSION

In an attempt to find a framework that combines aspects of general relativity and quantum theory, Oreshkov, Costa, and Brukner developed a formalism for quantum correlations with no causal order [5]. They proved that some achievable correlations are incompatible with any *a priori* causal order and, therefore, are not *causal realistic*. We consider the macroscopic limit of this framework and show that, in sharp contrast to the two-party scenario, macroscopic multi-party correlations surprisingly can be incompatible with any *a priori* causal order. To show this, we first propose a game that cannot be won in a causal realistic scenario, but is won with certainty when no causal order is fixed. It is left open whether a different game exists among more than two parties which re-opens the gap between the classical and the quantum success probability, and what additional assumptions to the quantum framework render its macroscopic limit causal realistic.

Recently, the idea of indefinite causal order was applied to quantum computation [7, 8]. Furthermore, Aaronson and Watrous showed that closed timelike curves render classical and quantum computing equivalent [9], a result similar to ours, in the sense that the winning probability of the causal game is the same for the quantum and for the classical framework. Closed timelike curves are an interpretation of the framework for correlations with no causal order, as the W object in Figure 2 could be thought of as a channel back in time, and are consistent with the theory of general relativity [10]. However, Aaronson and Watrous take Deutsch's approach [11] to closed timelike curves, which, as opposed to the framework studied here, is a non-linear extension of quantum theory.

Given the common world view that time and space are fundamental, quantum theory and all so far known interpretations (see Bell [12] for a review, or the parallel lives interpretation [13] as an example) fail in giving an explanatory description of nature, e.g., non-local correlations, leaving us with a mere mathematical formalism. Thus, it would be interesting to study theories that take space and time not as fundamental, but as emerging from other principles [14, 15] (this idea dates back around 2500 years [16]), as we believe they might help in understanding such correlations.

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diagonal in the computational basis, all local operations can be reduced to objects diagonal in the same basis. This work was supported by the Swiss National Science Foundation (SNF), the NCCR QSIT, and the COST action on Fundamental Problems in Quantum Physics.

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